SOME WEAK INDIVISIBILITY RESULTS IN ULTRAHOMOGENEOUS METRIC SPACES.

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ABSTRACT. We study the validity of a partition property known as weak indivisibility for the integer and the rational Urysohn metric spaces. We also compare weak indivisiblity to another partition property, called age-indivisibility, and provide an example of a countable ultrahomogeneous metric space which is age-indivisible but not weakly indivisible.

1. INTRODUCTION.

The purpose of this article is the study of certain partition properties of particular metric spaces, called *ultrahomogeneous*. A metric space is ultrahomogeneous when every isometry between finite metric subspaces of $\mathbf{X}$ can be extended to an isometry of $\mathbf{X}$ onto itself. For example, when seen as a metric space, any Euclidean space $\mathbb{R}^n$ has this property. So does the separable infinite dimensional Hilbert space ℓ_2 and its unit sphere $\mathbb{S}^\infty$. Another less known example of ultrahomogeneous metric space, though recently a well studied object (see [U08]), is the *Urysohn space*, denoted $\mathbf{U}$: up to isometry, it is the unique complete separable ultrahomogeneous metric space into which every separable metric space embeds (here and in the sequel, all the embeddings are *isometric*, that is, distance preserving). This space also admits numerous countable analogs. For example, for various countable sets S of positive reals (see [DLPS07] for the precise condition on S), there is, up to isometry, a unique countable ultrahomogeneous metric space into which every countable metric space with distances in S embeds. When $S = \mathbb{Q}$ or $\mathbb{N}$ this gives raise to the spaces denoted respectively $\mathbf{U}_\mathbb{Q}$ (the *rational Urysohn space*) and $\mathbf{U}_\mathbb{N}$ (the *integer Urysohn space*). Recently, separable ultrahomogeneous metric spaces have been at the center of active research because of a remarkable connection between their combinatorial behavior when submitted to finite partitions and the dynamical properties of their isometry group. For example, consider the following result. Call a metric space $\mathbf{Z} = (Z, d^{\mathbf{Z}})$ *age-indivisible* if for every finite metric subspace $\mathbf{Y}$ of $\mathbf{Z}$ and every partition $Z = B \cup R$ (thought as a coloring of the points of Z with two colors, blue and red), the space $\mathbf{Y}$ embeds in B or R.

Theorem (Folklore). *The spaces $\mathbf{U}_\mathbb{Q}$ and $\mathbf{U}_\mathbb{N}$ are age-indivisible.*

There are at least two directions for possible generalizations. First, one may ask what happens if instead of coloring the points of, say, the space $\mathbf{U}_\mathbb{Q}$, we color the isometric copies of a fixed finite metric subspace $\mathbf{X}$ of $\mathbf{U}_\mathbb{Q}$. We will not touch this subject here but Kechris, Pestov and Todorcevic showed in [KPT05] that the

answer to this question (obtained by Nešetřil in [N07]) has spectacular consequences on the groups iso($\mathbf{U}_\mathbb{Q}$) and iso($\mathbf{U}$) of surjective self-isometries of $\mathbf{U}_\mathbb{Q}$ and $\mathbf{U}$. For example, every continuous action of iso($\mathbf{U}$) (equipped with the pointwise convergence topology) on a compact topological space admits a fixed point.

Another direction of generalization is to ask whether any of those spaces is *indivisible*, that is, whether B or R necessarily contains not only a copy of a fixed finite $\mathbf{Y}$ but of the whole space itself. However, it is known that any indivisible metric space must have a bounded distance set. Therefore, the spaces $\mathbf{U}_\mathbb{Q}$ and $\mathbf{U}_\mathbb{N}$ are not indivisible. Still, in this article, we investigate whether despite this obstacle, a partition result weaker than indivisibility but stronger than age-indivisibility holds. Call a metric space $\mathbf{X}$ *weakly indivisible* when for every finite metric subspace $\mathbf{Y}$ of $\mathbf{X}$ and every finite partition $X = B \cup R$, either $\mathbf{Y}$ embeds in B or $\mathbf{X}$ embeds in R. Building on techniques developed in [LN08] and [NS-], we prove:

Theorem 1. *The space $\mathbf{U}_\mathbb{N}$ is weakly indivisible.*

As for $\mathbf{U}_\mathbb{Q}$, we are not able to prove or disprove weak indivisibility but we obtain the following weakening. If $\mathbf{X}$ is a metric space, $Y \subset X$ and $\varepsilon > 0$, $(Y)_\varepsilon$ denotes the set

$$(Y)_\varepsilon = \{x \in X : \exists y \in Y \quad d^{\mathbf{X}}(x, y) \leqslant \varepsilon\}.$$

Theorem 2. *Let $\mathbf{U}_\mathbb{Q} = B \cup R$ and $\varepsilon > 0$. Assume that there is a finite metric subspace $\mathbf{Y}$ of $\mathbf{U}_\mathbb{Q}$ that does not embed in B. Then $\mathbf{U}_\mathbb{Q}$ embeds in $(R)_\varepsilon$.*

This in turn leads to the following partition result for $\mathbf{U}$:

Theorem 3. *Let $\mathbf{U} = B \cup R$ and $\varepsilon > 0$. Assume that there is a compact metric subspace $\mathbf{K}$ of $\mathbf{U}$ that does not embed in $(B)_\varepsilon$. Then $\mathbf{U}$ embeds in $(R)_\varepsilon$.*

Note that those results do not answer the following: for a countable ultrahomogeneous metric space is weak indivisibility a strictly stronger property than age-indivisibility? In the last section of this paper, we answer that question by producing an example of countable ultrahomogeneous metric space which is age-indivisible but not weakly indivisible. To our knowledge, this is even one of the first two known examples of a countable ultrahomogeneous relational structure witnessing that weak indivisibility and age-indivisiblity are distinct properties (the other example will appear in [LNS-]). Let $\mathcal{E}_\mathbb{Q}$ be the class of all finite metric spaces $\mathbf{X}$ with distances in $\mathbb{Q}$ which embed isometrically into the unit sphere $\mathbb{S}^\infty$ of ℓ_2 with the property that $\{0_{\ell_2}\} \cup \mathbf{X}$ is affinely independent. It is known that there is a unique countable ultrahomogeneous metric space $\mathbb{S}^\infty_\mathbb{Q}$ whose class of finite metric spaces is exactly $\mathcal{E}_\mathbb{Q}$, and that the metric completion of $\mathbb{S}^\infty_\mathbb{Q}$ is $\mathbb{S}^\infty$ (for a proof, see [NVT06] or [NVT-]).

Theorem 4. *The space $\mathbb{S}^\infty_\mathbb{Q}$ is age-indivisible.*

Theorem 5. *The space $\mathbb{S}^\infty_\mathbb{Q}$ is not weakly indivisible.*

The proof of each of those results requires the use of a deep theorem: the proof of Theorem 7 is based on a central result of Matoušek and Rödl in Euclidean Ramsey theory, while the proof of Theorem 8 lies on a strong form of the Odell-Schlumprecht distortion theorem in Banach space theory.

The paper is organized as follows. In Section 2, we prove Theorem 1. In section 3, we prove Theorem 2. Theorem 3 is proved in Section 4, and Theorems 4 and 5 are proved in Section 5.

2. Proof of Theorem 1

The purpose of this section is to prove Theorem 1. In fact, we prove a slightly stronger result. We mentioned in introduction that there are various countable sets S of positive reals for which there is, up to isometry, a unique countable ultrahomogeneous metric space into which every countable metric space with distances in S embeds. It can be proved that when $p \in \mathbb{N}$, the integer interval $\{1, \ldots, p\}$ is such a set. The corresponding countable ultrahomogeneous metric space is denoted $\mathbf{U}_p$.

Theorem 6. *Let $\boldsymbol{U}_{\mathbb{N}} = B \cup R$. Assume that there is $p \in \omega$ such that $\boldsymbol{U}_p$ does not embed in B. Then $\boldsymbol{U}_{\mathbb{N}}$ embeds in R.*

The rest of this section is devoted to a proof of Theorem 6. We fix $p \in \mathbb{N}$ as well as a partition $\mathbf{U}_{\mathbb{N}} = B \cup R$ such that $\mathbf{U}_p$ does not embed in B. Our goal is to prove that $\mathbf{U}_{\mathbb{N}}$ embeds into R. Let $m := \lceil p/2 \rceil$ (the least integer larger or equal to $p/2$). Recall that if $Y \subset \mathbf{U}_{\mathbb{N}}$, the set $(Y)_\varepsilon$ is defined by

$$(Y)_\varepsilon = \{x \in X : \exists y \in Y \ \ d^{\mathbf{X}}(x, y) \leqslant \varepsilon\}.$$

In particular, if $x \in \mathbf{U}_{\mathbb{N}}$, the set $(\{x\})_{m-1}$ denotes the set of all elements of $\mathbf{U}_{\mathbb{N}}$ at distance $\leq m-1$ from x. We are going to construct $\widetilde{\mathbf{U}} \subset R$ isometric to $\mathbf{U}_{\mathbb{N}}$ recursively such that for every $x \in \widetilde{\mathbf{U}}$,

$$(\{x\})_{m-1} \cap \widetilde{\mathbf{U}} \subset R.$$

More precisely, fix an enumeration $\{x_n : n \in \mathbb{N}\}$ of $\mathbf{U}_{\mathbb{N}}$. We are going to construct $\{\tilde{x}_n : n \in \mathbb{N}\} \subset \mathbf{U}_{\mathbb{N}}$ recursively together with a decreasing sequence $(\mathbf{D}_n)_{n \in \mathbb{N}}$ of metric subspaces of $\mathbf{U}_{\mathbb{N}}$ such that $x_n \mapsto \tilde{x}_n$ is an isometry and, for every $n \in \mathbb{N}$, each $\mathbf{D}_n$ is isometric to $\mathbf{U}_{\mathbb{N}}$, $\{\tilde{x}_k : k \leq n\} \subset D_n$, and $(\{\tilde{x}_n\})_{m-1} \cap D_n \subset R$. To do so, we will need the notion of *Katětov* map as well as several technical lemmas.

Definition 1. *Given a metric space $\boldsymbol{X} = (X, d^{\boldsymbol{X}})$, a map $f : X \longrightarrow (0, +\infty)$ is* Katětov over $\mathbf{X}$ *when*

$$\forall x, y \in X, \ \ |f(x) - f(y)| \leqslant d^{\boldsymbol{X}}(x, y) \leqslant f(x) + f(y).$$

Equivalently, one can extend the metric $d^{\mathbf{X}}$ to $X \cup \{f\}$ by defining, for every x, y in X, $\widehat{d^{\mathbf{X}}}(x, f) = f(x)$ and $\widehat{d^{\mathbf{X}}}(x, y) = d^{\mathbf{X}}(x, y)$. The corresponding metric space is then written $\mathbf{X} \cup \{f\}$. The set of all Katětov maps over $\mathbf{X}$ is written $E(\mathbf{X})$. For a metric subspace $\mathbf{X}$ of $\mathbf{Y}$ and a Katětov map $f \in E(\mathbf{X})$, the *orbit of f in $\mathbf{Y}$* is the set $O(f, \mathbf{Y})$ defined by

$$O(f, \mathbf{Y}) = \{y \in Y : \forall x \in \mathbf{X} \ \ d^{\mathbf{Y}}(y, x) = f(x)\}.$$

Here, the concepts of Katětov map and orbit are relevant because of the following standard reformulation of the notion of ultrahomogeneity, which will be used in the sequel:

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Lemma 1. *Let $\boldsymbol{X}$ be a countable metric space. Then $\boldsymbol{X}$ is ultrahomogeneous iff for every finite subspace $\boldsymbol{F} \subset \boldsymbol{X}$ and every Katětov map f over $\boldsymbol{F}$, if $\boldsymbol{F} \cup \{f\}$ embeds into $\boldsymbol{X}$, then $O(f, \boldsymbol{X}) \neq \emptyset$.*

Proof. For a proof of that fact in the general context of relational structures, see for example [F00]. For a proof in the particular context of metric spaces, see [NVT06] or [NVT-]. □

Lemma 2. *Let G be a finite subset of $\boldsymbol{U}_{\mathbb{N}}$, and g a Katětov map with domain G and with values in $\mathbb{N}$. Then there exists an isometric copy $\boldsymbol{C}$ of $\boldsymbol{U}_{\mathbb{N}}$ inside $\boldsymbol{U}_{\mathbb{N}}$ such that:*

    (i) $G \subset C$,
    (ii) $O(g, \boldsymbol{C}) \subset B$ or $O(g, \boldsymbol{C}) \subset R$.

In words, Lemma 2 asserts that going to a subcopy of $\boldsymbol{U}_{\mathbb{N}}$ if necessary, we may assume that the orbit of g is completely included in one of the parts of the partition. Observe that as a metric space, the orbit $O(g, \mathbf{C})$ is isometric to $\mathbf{U}_n$ where $n = 2 \min g$ (Indeed, it is countable ultrahomogeneous with distances in $\{1, \ldots, n\}$ and embeds every countable metric space with distances in $\{1, \ldots, n\}$).

Proof. The proof of Lemma 2 can be found in [NS-]. More precisely, Lemma 2 can be obtained by combining Lemma 2 [NS-] and Lemma 3 [NS-] after having replaced $\mathbf{U}_p$ by $\mathbf{U}_{\mathbb{N}}$ in those statements. The proof of Lemma 3 [NS-] is elementary, while the proof of Lemma 2 [NS-] represents the core of [NS-]. Those two proofs can be carried out with no modification once $\mathbf{U}_p$ has been replaced by $\mathbf{U}_{\mathbb{N}}$. □

Lemma 3. *Let $G_0 \subset G$ be finite subsets of $\boldsymbol{U}_{\mathbb{N}}$, and let $\mathcal{G}$ a finite family of Katětov maps with domain G and such that for all $g, g' \in \mathcal{G}$:*

$$\max(|g - g'| \restriction G_0) = \max|g - g'|,$$
$$\min((g + g') \restriction G_0) = \min(g + g'),$$
$$\min(g \restriction G_0) = \min(g).$$

Then there exists an isometric copy $\boldsymbol{C}$ of $\boldsymbol{U}_{\mathbb{N}}$ inside $\boldsymbol{U}_{\mathbb{N}}$ such that:

    (i) $G \cap C = G_0$,
    (ii) $\forall g \in \mathcal{G} \quad O(g \restriction G_0, \boldsymbol{C}) \subset O(g, \boldsymbol{U}_{\mathbb{N}})$.

Proof. Lemma 3 is also a modified version of a result proved in [NS-], namely Lemma 5. Like in the case of Lemma 2, the proof of Lemma 5 [NS-] can be carried out without modification once $\mathbf{U}_p$ has been replaced by $\mathbf{U}_{\mathbb{N}}$. □

2.1. **Construction of $\tilde{x}_0$ and $\mathbf{D}_0$.** First, pick an arbitrary $u \in \mathbf{U}_{\mathbb{N}}$ and consider the map $g : \{u\} \longrightarrow \mathbb{N}$ defined by $g(u) = m$. By Lemma 2, find an isometric copy $\mathbf{C}$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{U}_{\mathbb{N}}$ such that:

    (i) $u \in C$,
    (ii) $O(g, \mathbf{C}) \subset B$ or $O(g, \mathbf{C}) \subset R$.

Note that since g has minimum m, the orbit $O(g, \mathbf{C})$ is isometric to $\mathbf{U}_{2m}$ and therefore contains a copy of $\mathbf{U}_p$. Hence, because $\mathbf{U}_p$ does not embed in B, the inclusion $O(g, \mathbf{C}) \subset B$ is excluded and we really have $O(g, \mathbf{C}) \subset R$. Let $\tilde{x}_0 \in O(g, \mathbf{C})$ and for every $k \leq m$ let $g_k : \{u, \tilde{x}_0\} \longrightarrow \mathbb{N}$ be such that $g_k(u) = m$ and $g_k(\tilde{x}_0) = k$. The sets $G_0 = \{\tilde{x}_0\}$ and $G = \{u, \tilde{x}_0\}$, and the family $\mathcal{G} = \{g_k : k \leq m\}$ satisfy the hypotheses of Lemma 3, which allows to obtain an isometric copy $\mathbf{D}_0$ of $\mathbf{U}_{\mathbb{N}}$ inside $\mathbf{C}$ such that:

(i) $\{u, \tilde{x}_0\} \cap D_0 = \{\tilde{x}_0\}$,
(ii) $\forall k \le m \;\; O(g_k \restriction \{\tilde{x}_0\}, \mathbf{D}_0) \subset O(g_k, \mathbf{C})$.

Note that for every $k \le m$, we have $O(g_k, \mathbf{C}) \subset O(g, \mathbf{C}) \subset R$. Therefore, in $\mathbf{D}_0$, all the spheres around $\tilde{x}_0$ with radius $k \le m$ are included in R. So

$$(\{\tilde{x}_0\})_{m-1} \cap D_0 \subset R. \qquad \square$$

2.2. **Induction step.** Assume that we constructed $\{\tilde{x}_k : k \le n\} \subset \mathbf{U}_\mathbb{N}$ together with a decreasing sequence $(\mathbf{D}_k)_{k \le n}$ of metric subspaces of $\mathbf{U}_\mathbb{N}$ such that $x_k \mapsto \tilde{x}_k$ is an isometry (recall that $\{x_n : n \in \mathbb{N}\}$ is the enumeration of $\mathbf{U}_\mathbb{N}$ we fixed at the beginning of the proof), each $\mathbf{D}_k$ is isometric to $\mathbf{U}_\mathbb{N}$, $\{\tilde{x}_k : k \le n\} \subset D_n$ and $(\{\tilde{x}_k\})_{m-1} \cap D_n \subset R$ for every $k \le n$. We are going to construct $\tilde{x}_{n+1}$ and $\mathbf{D}_{n+1}$. Consider the map $f : \{\tilde{x}_0, \ldots, \tilde{x}_n\} \longrightarrow \mathbb{N}$ where

$$\forall k \le n \;\; f(\tilde{x}_k) = d^{\mathbf{U}_\mathbb{N}}(x_k, x_{n+1}).$$

Consider the set $\mathcal{G}$ defined by

$$\{g \in E(\{\tilde{x}_0, \ldots, \tilde{x}_n\}) : \forall k \le n \; (|f(\tilde{x}_k - g(\tilde{x}_i))| \le m-1 \;\; \text{and} \;\; g(\tilde{x}_k) \ge m)\}.$$

This set is finite and a repeated application of Lemma 2 allows to construct an isometric copy $\mathbf{C}$ of $\mathbf{U}_\mathbb{N}$ inside $\mathbf{U}_\mathbb{N}$ such that:

(i) $\{\tilde{x}_0, \ldots, \tilde{x}_n\} \subset C$,
(ii) $\forall g \in \mathcal{G}, \;\; O(g, \mathbf{C}) \subset B$ or R.

Note that since every $g \in \mathcal{G}$ has minimum m, the orbit $O(g, \mathbf{C})$ is isometric to $\mathbf{U}_{2m}$ and therefore contains a copy of $\mathbf{U}_p$. Because $\mathbf{U}_p$ does not embed in B, we consequently have

$$\forall g \in \mathcal{G}, \;\; O(g, \mathbf{C}) \subset R.$$

Let $\tilde{x}_{n+1} \in O(f, \mathbf{C})$. We claim that $\tilde{x}_{n+1}$ is as required. Note that, because $\tilde{x}_{n+1} \in O(f, \mathbf{C})$, we have

$$\forall k \le n \;\; d^{\mathbf{U}_\mathbb{N}}(\tilde{x}_{n+1}, \tilde{x}_k) = f(\tilde{x}_k) = d^{\mathbf{U}_\mathbb{N}}(x_k, x_{n+1}).$$

Therefore, $x_k \mapsto \tilde{x}_k$ is an isometry. Next we prove that $(\{\tilde{x}_{n+1}\})_{m-1} \subset R$. Indeed, let $y \in (\{\tilde{x}_{n+1}\})_{m-1}$. If $d^{\mathbf{U}_\mathbb{N}}(\tilde{x}_k, y) \ge m$ for every $k \le n$, then the map $d^{\mathbf{U}_\mathbb{N}}(\cdot, y)$ is in $\mathcal{G}$ and so $y \in O(d^{\mathbf{U}_\mathbb{N}}(\cdot, y), \mathbf{C}) \subset R$. Otherwise, we have $d^{\mathbf{U}_\mathbb{N}}(\tilde{x}_k, y) \le m$ for some $k \le n$ and $y \in (\{\tilde{x}_k\})_{m-1} \subset R$. $\qquad \square$

3. Proof of Theorem 2

The purpose of this section is to prove Theorem 2. The main ingredients of the proofs are the result of Theorem 1 as well as the following technical lemma:

Lemma 4. *Let $q \in \mathbb{N}$ be positive. Then there is an isometric copy $\mathbf{U}^*_{\mathbb{N}/q}$ of $\mathbf{U}_{\mathbb{N}/q}$ in $\mathbf{U}_\mathbb{Q}$ such that for every subspace $\widetilde{V}$ of $\mathbf{U}^*_{\mathbb{N}/q}$ isometric to $\mathbf{U}_{\mathbb{N}/q}$, the set $(\tilde{V})_{1/2q}$ includes an isometric copy of $\mathbf{U}_\mathbb{Q}$.*

Proof. Lemma 4 is a modified version of a result proved in [LN08], whose statement appears at the very beginning of Proposition 5. Its proof is an easy modification of Lemma 2 [LN08] and is not included here. $\qquad \square$

Proof of Theorem 2. Choose $q \in \mathbb{N}$ large enough so that all distances appearing in $\mathbf{Y}$ are integer multiples of $1/q$ and $1/2q \leq \varepsilon$. The partition $\mathbf{U}_{\mathbb{Q}} = B \cup R$ induces a partition of $\mathbf{U}_{\mathbb{N}/q}^*$ (the space constructed in Lemma 4) where $\mathbf{Y}$ does not embed in B. By weak indivisibility of $\mathbf{U}_{\mathbb{N}}$, the space $\mathbf{U}_{\mathbb{N}/q}$ is weakly indivisible as well and so there is a subspace $\widetilde{V}$ of $\mathbf{U}_{\mathbb{N}/q}^*$ isometric to $\mathbf{U}_{\mathbb{N}/q}$ such that $\tilde{V} \subset R$. By construction of $\mathbf{U}_{\mathbb{N}/q}^*$, the set $(\tilde{V})_{1/2q}$ includes an isometric copy $\widetilde{U}$ of $\mathbf{U}_{\mathbb{Q}}$. Notice that $\tilde{U} \subset (\tilde{V})_{1/2q} \subset (\tilde{V})_{\varepsilon} \subset (R)_{\varepsilon}$. $\qquad\square$

4. Proof of Theorem 3

The purpose of this section is to prove Theorem 3. As for Theorem 2, we will use the result of Theorem 1 as well as several technical lemmas. The first one can be seen as a version of Lemma 4 in the context of the space $\mathbf{U}$:

Lemma 5. *Let $q \in \mathbb{N}$ be positive. Then there is an isometric copy $\mathbf{U}_{\mathbb{N}/q}^{**}$ of $\mathbf{U}_{\mathbb{N}/q}$ in $\mathbf{U}$ such that for every subspace $\widetilde{V}$ of $\mathbf{U}_{\mathbb{N}/q}^*$ isometric to $\mathbf{U}_{\mathbb{N}/q}$, the set $(\tilde{V})_{1/2q}$ includes an isometric copy of $\mathbf{U}$.*

Proof. Lemma 5 is a direct consequence of Lemma 4 and of the fact that $\mathbf{U}$ is the metric completion of $\mathbf{U}_{\mathbb{Q}}$. $\qquad\square$

The second lemma we will need states that in $\mathbf{U}$, the copies of the compact space $\mathbf{K}$ can be captured by a single finite metric subspace of $\mathbf{U}$:

Lemma 6. *There is a finite metric space $\mathbf{Y}$ of $\mathbf{U}$ with rational distances such that $\mathbf{K}$ embeds in $(\tilde{Y})_{\varepsilon}$ for every subspace $\widetilde{Y}$ of $\mathbf{U}$ isometric to $\mathbf{Y}$.*

Proof. Using compactness of $\mathbf{K}$, find a finite subspace $\mathbf{Z}$ of $\mathbf{K}$ such that $K \subset (Z)_{\varepsilon/2}$.

Claim 1. *The space $\mathbf{K}$ embeds in $(\tilde{Z})_{\varepsilon}$ for every subspace $\widetilde{Z}$ of $\mathbf{U}$ isometric to $\mathbf{Z}$.*

Proof. This follows from ultrahomogeneity of $\mathbf{U}$: if $\widetilde{Z}$ is a subspace of $\mathbf{U}$ isometric to $\mathbf{Z}$, let $\phi : Z \longrightarrow \tilde{Z}$ be an isometry. By ultrahomogeneity of $\mathbf{U}$, find $\Phi : \mathbf{U} \longrightarrow \mathbf{U}$ extending ϕ. Then $\Phi(K)$ is isometric to $\mathbf{K}$ and is included in

$$\Phi((Z)_{\varepsilon/2}) = (\Phi(Z))_{\varepsilon/2} = (\tilde{Z})_{\varepsilon/2}. \qquad\square$$

Therefore, the space $\mathbf{Z}$ is almost as required except that it may not have rational distances. To arrange that, consider $q \in \mathbb{N}$ large enough so that $1/q < \varepsilon/2$. For a number α, let $\lceil\alpha\rceil_q$ denote the smallest number $\geq \alpha$ of the form l/q with l integer. The function $\lceil\cdot\rceil_q$ is subadditive and increasing. Hence, the composition $\lceil d^{\mathbf{Z}}\rceil_q = \lceil\cdot\rceil_q \circ d^{\mathbf{Z}}$ is a metric on Z. Let $\mathbf{Y}$ be defined as the metric space $(Z, \lceil d^{\mathbf{Z}}\rceil_q)$. It obviously has rational distances. We are going to show that it is as required. Consider the set $X = Z \times \{0, 1\}$ and define

$$\delta((z, i), (z', i')) = \begin{cases} d^{\mathbf{Z}}(z, z') & \text{if } i = i' = 0, \\ \lceil d^{\mathbf{Z}}(z, z')\rceil_q & \text{if } i = i' = 1, \\ d^{\mathbf{Z}}(z, z') + \varepsilon/2 & \text{if } i \neq i'. \end{cases}$$

In spirit, the structure (X, δ) is obtained by putting a copy of $\mathbf{Y}$ $(= (Z, \lceil d^{\mathbf{Z}}\rceil_q))$ above a copy of $\mathbf{Z}$ such that the distance between any point $(z, 0) \in Z \times \{0\}$ and its counterpart $(z, 1)$ in $Z \times \{1\}$ is $\varepsilon/2$.

Claim 2. *The map δ is a metric on X.*

Proof. The maps $d^{\mathbf{Z}}$ and $\lceil d^{\mathbf{Z}} \rceil_q$ being metrics on $Z \times \{0\}$ and $Z \times \{1\}$, it suffices to verify that the triangle inequality is satisfied on triangles of the form $\{(x,0),(y,0),(z,1)\}$ and $\{(x,1),(y,1),(z,0)\}$, with $x,y,z \in Z$.

Assume that $x,y,z \in Z$, and consider the triangle $\{(x,1),(y,1),(z,0)\}$. Then

$$\delta((x,1),(z,0)) = d^{\mathbf{Z}}(x,z) + \frac{\varepsilon}{2}$$
$$\leq d^{\mathbf{Z}}(x,y) + d^{\mathbf{Z}}(y,z) + \frac{\varepsilon}{2}$$
$$\leq \lceil d^{\mathbf{Z}}(x,y) \rceil_q + d^{\mathbf{Z}}(y,z) + \frac{\varepsilon}{2}$$
$$\leq \delta((x,1),(y,1)) + \delta((y,1),(z,0)).$$

Similarly,

$$\delta((y,1),(z,0)) \leq \delta((y,1),(x,1)) + \delta((x,1),(z,0)).$$

And finally,

$$\delta((x,1),(y,1)) = \lceil d^{\mathbf{Z}}(x,y) \rceil_q$$
$$\leq d^{\mathbf{Z}}(x,y) + \frac{1}{q}$$
$$\leq d^{\mathbf{Z}}(x,y) + \frac{\varepsilon}{2}$$
$$\leq d^{\mathbf{Z}}(x,z) + d^{\mathbf{Z}}(z,y) + \frac{\varepsilon}{2}$$
$$\leq d^{\mathbf{Z}}(x,z) + \frac{\varepsilon}{2} + d^{\mathbf{Z}}(z,y) + \frac{\varepsilon}{2}$$
$$\leq \delta((x,1),(z,0)) + \delta((z,0),(y,1)).$$

Next, consider the triangle $\{(x,0),(y,0),(z,1)\}$. We have

$$\delta((x,0),(z,1)) = d^{\mathbf{Z}}(x,z) + \frac{\varepsilon}{2}$$
$$\leq d^{\mathbf{Z}}(x,y) + d^{\mathbf{Z}}(y,z) + \frac{\varepsilon}{2}$$
$$\leq \delta((x,0),(y,0)) + \delta((y,0),(z,1)).$$

Similarly,

$$\delta((y,0),(z,1)) \leq \delta((y,0),(x,0)) + \delta((x,0),(z,1)).$$

Finally,

$$\delta((x,0),(y,0)) = d^{\mathbf{Z}}(x,y)$$
$$\leq d^{\mathbf{Z}}(x,z) + d^{\mathbf{Z}}(z,y)$$
$$\leq d^{\mathbf{Z}}(x,z) + \frac{\varepsilon}{2} + d^{\mathbf{Z}}(z,y) + \frac{\varepsilon}{2}$$
$$\leq \delta((x,0),(z,1)) + \delta((z,1),(y,0)). \qquad \square$$

Denote the space (X,δ) by $\mathbf{X}$. Recall that every finite metric space embeds isometrically in $\mathbf{U}$. Hence, without loss of generality, we may suppose $Y \subset X \subset \mathbf{U}$. We claim that $\mathbf{Y}$ is as required. By construction, the space $\mathbf{Y}$ is a finite subspace of $\mathbf{U}$ with rational distances. Observe that $X \subset (Y)_{\varepsilon/2}$. Assume that a subspace $\widetilde{\mathbf{Y}}$ of $\mathbf{U}$ is isometric to $\mathbf{Y}$. By an argument similar to the one used in Claim 1, the space $\mathbf{X}$ embeds in $(\tilde{Y})_{\varepsilon/2}$. Thus, because $\mathbf{Z}$ embeds in $\mathbf{X}$, the set $(\tilde{Y})_{\varepsilon/2}$ contains

a copy of $\mathbf{Z}$, call it $\widetilde{\mathbf{Z}}$. By Claim 1, the set $(\tilde{Z})_\varepsilon$ contains a copy of $\mathbf{K}$, call it $\widetilde{\mathbf{K}}$. Then

$$\tilde{K} \subset (\tilde{Z})_\varepsilon \subset ((\tilde{Y})_{\varepsilon/2})_{\varepsilon/2} \subset (\tilde{Y})_\varepsilon.$$

This finishes the proof of Lemma 6. $\qquad\square$

Proof of Theorem 3. Choose $q \in \mathbb{N}$ large enough so that $1/2q \leq \varepsilon$ and all distances appearing in $\mathbf{Y}$ are integer multiples of $1/q$. The partition $\mathbf{U} = B \cup R$ induces a partition of $\mathbf{U}_{\mathbb{N}/q}^{**}$ provided by Lemma 5. Note that $\mathbf{Y}$ does not embed in B: indeed, if a subspace $\widetilde{\mathbf{Y}}$ of B were isometric to $\mathbf{Y}$, then $(\tilde{Y})_\varepsilon \subset (B)_\varepsilon$ and by Lemma 6, the space $\mathbf{K}$ would embed in $(B)_\varepsilon$, which is not the case. Therefore, by weak indivisibility of $\mathbf{U}_{\mathbb{N}/q}$, there is a subspace $\widetilde{\mathbf{V}}$ of $\mathbf{U}_{\mathbb{N}/q}^*$ isometric to $\mathbf{U}_{\mathbb{N}/q}$ such that $\tilde{V} \subset R$. By construction of $\mathbf{U}_{\mathbb{N}/q}^*$, the set $(\tilde{V})_{1/2q}$ includes an isometric copy $\widetilde{\mathbf{U}}$ of $\mathbf{U}$. Notice that $\tilde{U} \subset (\tilde{V})_{1/2q} \subset (\tilde{V})_\varepsilon \subset (R)_\varepsilon$. $\qquad\square$

5. Age-indivisibility does not imply weak indivisibility

In this section, we prove results that are slightly stronger than Theorems 7 and 8. In what follows, the set S is a fixed dense subset of $[0, 2]$. Let $\mathcal{E}_S$ be the class of all finite metric spaces $\mathbf{X}$ with distances in S which embed isometrically into the unit sphere $\mathbb{S}^\infty$ of ℓ_2 with the property that $\{0_{\ell_2}\} \cup \mathbf{X}$ is affinely independent.

Claim 3. *There is a unique countable ultrahomogeneous metric space $\mathbb{S}_S^\infty$ whose class of finite metric spaces is exactly $\mathcal{E}_S$. Moreover, the metric completion of $\mathbb{S}_S^\infty$ is $\mathbb{S}^\infty$.*

Proof. See [NVT06] or [NVT-]. $\qquad\square$

We show:

Theorem 7. *The space $\mathbb{S}_S^\infty$ is age-indivisible.*

Theorem 8. *The space $\mathbb{S}_S^\infty$ is not weakly indivisible.*

The proof of those results are provided in Subsection 5.1 and Subsection 5.2 respectively.

5.1. The space $\mathbb{S}_S^\infty$ is age-indivisible. Let $\mathbf{Y}$ be a finite metric subspace of $\mathbb{S}_S^\infty$. We need to show:

Claim 4. *There is a finite metric subspace $\mathbf{Z}$ of $\mathbb{S}_S^\infty$ such that for every partition $Z = B \cup R$, the space $\mathbf{Y}$ embeds in B or R.*

The main ingredient of the proof is the following deep result due to Matoušek and Rödl:

Theorem 9 (Matoušek-Rödl [MR95])**.** *Let $\mathbf{X}$ be an affinely independent finite metric subspace of $\mathbb{S}^\infty$ with circumradius r, and let $\alpha > 0$. Then there is a finite metric subspace $\mathbf{Z}$ of $\mathbb{S}^\infty$ with circumradius $r + \alpha$ such that for every partition $Z = B \cup R$, the space $\mathbf{X}$ embeds in B or R.*

What we need to prove is that in the case where $\mathbf{X} = \mathbf{Y}$, we may arrange $\mathbf{Z}$ to be a subspace of $\mathbb{S}_S^\infty$ (that is, with distances in S and $\{0_{\ell_2}\} \cup \mathbf{Z}$ affinely independent). We will make use of the following facts along the way:

Theorem 10 (Schoenberg [S38])**.** *Let $X = \{x_k : 1 \le k \le |G|\}$ be a finite set and let $\delta : X^2 \longrightarrow \mathbb{R}$ satisfying:*

(i) for every $x \in X$, $\delta(x, x) = 0$,

(ii) for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$.

Then (X, δ) is isometric to a subset of ℓ_2 iff

$$\max \left\{ \sum_{1 \le i < j \le n} \delta(x_i, x_j)^2 x_i x_j : \sum_{k=1}^{n} x_k^2 = 1 \ \ and \ \ \sum_{k=1}^{n} x_k = 0 \right\} \le 0 \ .$$

Moreover, (X, δ) is isometric to an affinely independent subset of ℓ_2 iff the inequality is strict.

Claim 5. *Let $\mathbf{X}$ be a finite affinely independent metric subspace of $\mathbb{S}^\infty$ with circumradius r. Then there is $\varepsilon > 0$ such that for every $\delta : X^2 \longrightarrow \mathbb{R}$ satisfying*

(i) for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$,

(ii) $|\delta - d^{\mathbf{X}}| < \varepsilon$,

the space (X, δ) is an affinely independent metric subspace of $\mathbb{S}^\infty$.

Proof. Direct from Theorem 10 and from the fact that the map $M \mapsto Q_M$ is continuous, where for a matrix $M = (m_{ij})_{1 \le i, j \le n}$,

$$Q_M = \max \left\{ \sum_{1 \le i < j \le n} m_{ij} x_i x_j : \sum_{k=1}^{n} x_k^2 = 1 \ \ and \ \ \sum_{k=1}^{n} x_k = 0 \right\}. \qquad \square$$

Claim 6. *Let $\mathbf{X}$ be a finite metric subspace of $\mathbb{S}^\infty$ with circumradius r. Let $\varepsilon > 0$. Then $(X, d^{\mathbf{X}} + \varepsilon)$ is Euclidean, affinely independent with circumradius at most $r + \varepsilon$.*

Proof. Let V be the affine space spanned by X. Choose $(e_x)_{x \in X}$ a family of pairwise orthogonal vectors in $V^\perp$. For $x \in X$, set $\tilde{x} = x + \sqrt{\varepsilon/2}\, e_x$. Then the set $\{\tilde{x} : x \in X\}$ is affinely independent and is isometric to $(X, d^{\mathbf{X}} + \varepsilon)$. Its circumradius is at most $r + \varepsilon$ because it is contained in the ball centered at the circumcenter of X and with radius $r + \varepsilon$. $\qquad \square$

Claim 7. *Let $\mathbf{X}$ be an affinely independent subspace of $\mathbb{S}^\infty$. Then $\mathbf{X} \cup \{0_{\ell_2}\}$ is affinely independent iff the circumradius of $\mathbf{X}$ is < 1.*

Proof. Let V be the affine space spanned by X. Then the set $\mathbb{S}^\infty \cap V$ is the circumscribed sphere of X in V. It has radius < 1 iff 0_{ℓ_2} does not belong to V. $\qquad \square$

Proof of Claim 4. First, we show that there is an affinely independent finite metric subspace $\mathbf{Z}_0$ of $\mathbb{S}^\infty$ with circumradius < 1 such that for every partition $Z_0 = B \cup R$, $\mathbf{Y}$ embeds in B or R:

Let r denote the circumradius of $\mathbf{Y}$. Because $\mathbf{Y}$ is a subspace of $\mathbb{S}_S^\infty$, the space $\mathbf{Y} \cup \{0_{\ell_2}\}$ is affinely independent and by Claim 7, we have $r < 1$. By Claim 5, fix $\varepsilon > 0$ such that $r + 2\varepsilon < 1$ and such that for every map $\delta : X^2 \longrightarrow \mathbb{R}$ satisfying

(i) for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$,

(ii) $|\delta - d^{\mathbf{X}}| < \varepsilon$,

the space (Y, δ) is still Euclidean and affinely independent. Fix $\alpha > 0$ such that $\alpha < \varepsilon$. By choice of ε, the space $(Y, d^{\mathbf{Y}} - \varepsilon)$ is still Euclidean and affinely independent. It should be clear that its circumradius is at most r. Apply Theorem 9 to produce a finite metric subspace $\mathbf{T}$ of $\mathbb{S}^\infty$ with circumradius $r + \alpha$ such that

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for every partition $T = B \cup R$, the space $(Y, d^{\mathbf{Y}} - \varepsilon)$ embeds in B or R. Set $\mathbf{Z}_0 = (T, d^{\mathbf{T}} + \varepsilon)$. We claim that $\mathbf{Z}_0$ is as required.

Indeed, by Claim 6, $\mathbf{Z}_0$ is Euclidean, affinely independent, and its circumradius is at most $r + \alpha + \varepsilon < r + 2\varepsilon < 1$. Next, if $Z_0 = B \cup R$, this partition induces a partition $T = B \cup R$. By construction of $\mathbf{T}$, there is a subspace $\widetilde{\mathbf{Y}}$ of $\mathbf{T}$ isometric to $(Y, d^{\mathbf{Y}} - \varepsilon)$ contained in B or R. Note that in $\mathbf{Z}_0$, the metric subspace supported by $\widetilde{\mathbf{Y}}$ is isometric to $(Y, d^{\mathbf{Y}} - \varepsilon + \varepsilon) = \mathbf{Y}$.

Consider the space $\mathbf{Z}_0$ we just constructed. Using Claim 5 as well as the denseness of S, we may modify slightly all the distances in $\mathbf{Z}_0$ that are not in S and turn $\mathbf{Z}_0$ into an affinely independent subspace $\mathbf{Z}$ of $\mathbb{S}^\infty$ with distances in S and circumradius < 1. By Claim 7, the space $\{0_{\ell_2}\} \cup \mathbf{Z}$ is affinely independent. Therefore, $\mathbf{Z}$ embeds in $\mathbb{S}_S^\infty$. Finally, note that since all the distances of $\mathbf{Z}_0$ that were already in S did not get changed, the copies of $\mathbf{Y}$ in $\mathbf{Z}_0$ remain unaltered when passing to $\mathbf{Z}$. It follows that for every partition $Z = B \cup R$, the space $\mathbf{Y}$ embeds in B or R. $\qquad\square$

5.2. The space $\mathbb{S}_S^\infty$ is not weakly indivisible. The starting point of our proof of Theorem 8 is the following theorem:

Theorem 11 (Odell-Schlumprecht [OS94]). *There is a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that*

    (i) For every linear subspace V of ℓ_2 with $\dim V = \infty$, $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$.
    (ii) For every linear subspace V of ℓ_2 with $\dim V = \infty$, $\mathbb{S}^\infty \cap V \not\subset (R)_\varepsilon$.

In response to an inquiry of the authors, Thomas Schlumprecht [S08] indicated that the method that was used to prove Theorem 11 in [OS94] (where the statement is proved first in another Banach space known as the Schlumprecht space, and then transferred to ℓ_2), can be adapted to show that $\dim V = \infty$ may be replaced by $\dim V = 2$ in (i):

Theorem 12 (Odell-Schlumprecht). *There is a partition $\mathbb{S}^\infty = B \cup R$ and $\varepsilon > 0$ such that*

    (i) For every linear subspace V of ℓ_2 with $\dim V = 2$, $\mathbb{S}^\infty \cap V \not\subset (B)_\varepsilon$.
    (ii) For every linear subspace V of ℓ_2 with $\dim V = \infty$, $\mathbb{S}^\infty \cap V \not\subset (R)_\varepsilon$.

We are going to show how this result almost directly leads to Theorem 8. Consider the partition of $\mathbb{S}^\infty$ provided by Theorem 12. It should be clear that it induces a partition of $\mathbb{S}_S^\infty$.

Claim 8. $\mathbb{S}_S^\infty = B \cup R$ *witnesses that $\mathbb{S}_S^\infty$ is not weakly indivisible.*

The proof makes use of the following fact, which we prove for completeness:

Claim 9. *Let $Y \subset \mathbb{S}^\infty$ be isometric to $\mathbb{S}^\infty$. Then there is a closed linear subspace V of ℓ_2 with $\dim V = \infty$ such that $\bar{Y} = V \cap \mathbb{S}^\infty$.*

Proof. Consider V the closed linear span of Y in ℓ_2. Consider also the set $W = \{\lambda y : \lambda \in \mathbb{R}, \ y \in Y\}$. We will be done if we show $V = W$. Clearly, $W \subset V$. For the reverse inclusion, observe that because Y is closed (it is isometric to a complete metric space), the set W is closed. Therefore, it is enough to show that all the finite linear combinations of elements of V that have norm 1 are in Y, ie for every $y_1, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^n \lambda_i y_i \neq 0_{\ell_2}$,

$$\frac{\sum_{i=1}^n \lambda_i y_i}{\|\sum_{i=1}^n \lambda_i y_i\|} \in Y.$$

We proceed by induction on n. For $n = 2$, we first consider the case $\lambda_1 = \lambda_2 = 1$. Note that y_1 and y_2 cannot be antipodal (otherwise $y_1 + y_2 = 0_{\ell_2}$), and that $\frac{y_1+y_2}{\|y_1+y_2\|}$ can be characterized metrically in terms of y_1 and y_2. For example, it it is the unique geodesic middle point of y_1 and y_2 in the intrinsic metric on $\mathbb{S}^\infty$. Since the intrinsic metric can be defined in terms of the usual Hilbertian metric on $\mathbb{S}^\infty$, this point must belong to Y. By a usual middle-point-type argument, it follows that the entire geodesic segment between y_1 and y_2 is contained in Y. Using then that Y is closed under antipodality (because Y being isometric to $\mathbb{S}^\infty$ any $y \in Y$ must have a point at distance 2), as well as a middle-point-type argument again, the entire great circle through y_1 and y_2 is contained in Y. That finishes the case $n = 2$. Assume that the property is proved up to $n \geq 2$. Fix $y_1, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then writing

$$z = \frac{\sum_{i=1}^{n} \lambda_i y_i}{\|\sum_{i=1}^{n} \lambda_i y_i\|} \ ,$$

the vector

$$\frac{\sum_{i=1}^{n+1} \lambda_i y_i}{\left\|\sum_{i=1}^{n+1} \lambda_i y_i\right\|}$$

is a linear combination of z and y_{n+1} with norm 1. Therefore, it is of the form

$$\frac{\alpha z + \beta y_{n+1}}{\|\alpha z + \beta y_{n+1}\|}.$$

By induction hypothesis, z is in Y. So again by induction hypothesis (case $n = 2$),

$$\frac{\alpha z + \beta y_{n+1}}{\|\alpha z + \beta y_{n+1}\|} \in Y.$$

Therefore,

$$\frac{\sum_{i=1}^{n+1} \lambda_i y_i}{\left\|\sum_{i=1}^{n+1} \lambda_i y_i\right\|} \in Y. \hspace{2cm} \square$$

Proof of Claim 8. Let W be a linear subspace of ℓ_2 with $\dim W = 2$. By compactness of $\mathbb{S}^\infty \cap W$ and denseness of $\mathbb{S}_S^\infty$ in $\mathbb{S}^\infty$, there is $X \subset \mathbb{S}_S^\infty$ finite such that $\mathbb{S}^\infty \cap W \subset (X)_\varepsilon$. Let $\mathbf{X}$ denote the metric subspace of $\mathbb{S}_S^\infty$ supported by the set X. Then $\mathbf{X}$ does not embed in B because otherwise, there would be a linear subspace V of ℓ_2 with $\dim V = 2$ such that $\mathbb{S}^\infty \cap V \subset (B)_\varepsilon$, violating (i) of Theorem 12. On the other hand, $\mathbb{S}_S^\infty$ cannot embed in R: let $Y \subset \mathbb{S}_S^\infty$ be isometric to $\mathbb{S}_S^\infty$. Then in $\mathbb{S}^\infty$, the closure $\bar{Y}$ of Y is isometric to $\mathbb{S}^\infty$. By Claim 8, there is a closed linear subspace V of ℓ_2 with $\dim V = \infty$ such that $\bar{Y} = V \cap \mathbb{S}^\infty$. By (ii) of Theorem 12, $\bar{Y} \not\subset (R)_\varepsilon$. Therefore $\bar{Y} \not\subset R$. $\hspace{1cm} \square$

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